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The second pluri-genus of surface singularities

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1 Preliminary

Let (X, x) be a normal n -dimensional isolated singularity over \mathbb{C} and $f : (M, E) \rightarrow (X, x)$ a resolution of the singularity (X, x) with exceptional locus $E = f^{-1}(x)$. We say a resolution f is *good* if E is a divisor of normal crossings. Then the m -th L^2 -*plurigenus* of (X, x) is an integer $\delta_m(X, x)$ which was introduced in [5] and can be computed as

$$\delta_m(X, x) = \dim_{\mathbb{C}} H^0(M - E, \mathcal{O}_M(mK)) / H^0(M, \mathcal{O}_M(mK + (m-1)E)),$$

where K denotes the canonical divisor on M .

The invariant $p_g(X, x)$ called the *geometric genus* of a singularity (X, x) is defined by

$$p_g(X, x) = \dim_{\mathbb{C}} (R^{n-1} f_* \mathcal{O}_M)_x.$$

We note that $p_g(X, x) = \delta_1(X, x)$.

Normal surface singularities are classified as the following theorem.

(1.1) THEOREM[1, Theorem 5.4]. *For a normal surface singularity (X, x) , we have the following:*

(1) $\delta_m(X, x) = 0$ for all $m \in \mathbb{N}$, when (X, x) is a quotient singularity;
 (2) $\delta_m(X, x) \leq 1$ for all $m \in \mathbb{N}$ and $\delta_n(X, x) = 1$ for some $n \in \mathbb{N}$, in one of the following cases;

(a) (X, x) is a simple elliptic singularity;
 (b) (X, x) is a cusp singularity;
 (c) (X, x) is the quotient by a finite group of a singularity of type (a) or (b);

(3) $0 < \limsup_{m \rightarrow \infty} \delta_m(X, x)/m^2 < \infty$, in other cases.

(1.2) DEFINITION. A normal surface singularity (X, x) is of *general type* if which is in the class (3) above.

(1.3) DEFINITION. A normal surface singularity (X, x) is *minimally elliptic* if $p_g(X, x) = 1$ and (X, x) is a Gorenstein singularity.

2 The minimally elliptic singularities

Throughout this section, we assume that (X, x) is a minimally elliptic singularity, $f : (M, E) \rightarrow (X, x)$ the minimal good resolution with the irreducible decomposition $E = \bigcup_{i=1}^k E_i$ and K the canonical divisor on M

which is supported on E .

Then $-K \geq E$, and $-K = E$ if and only if (X, x) is a simple elliptic or a cusp singularity. If (X, x) is not a simple elliptic singularity, then $E_i \cong \mathbf{P}^1$ for all i .

(2.1) DEFINITION. Let Z be the fundamental cycle on M . We say the dual graph of E is *obtained* from another singularity (X', x') if the self-intersection number of the fundamental cycle of (X', x') is -1 and the weighted dual graphs of (X, x) and (X', x') are same except for self-intersection numbers of the components of E with multiplicity 1 in Z .

In [6], we have followings.

(2.2) THEOREM. For a minimally elliptic singularity (X, x) ,

$$\delta_2(X, x) = \dim_{\mathbf{C}} H_E^1(M, \mathcal{O}_M(2K + E)).$$

If (X, x) is of general type, $\delta_2(X, x) = KD + 2$, where $D = -K - E$.

(2.3) COROLLARY. If (X, x) is a hypersurface (resp. complete intersection), then $\delta_2(X, x) \leq 4$ (resp. 5).

(2.4) PROPOSITION. Let (X, x) be a minimally elliptic singularity of general type. Then $\delta_2(X, x) = 1$ (resp. 2) if and only if (X, x) is obtained from a unimodal (resp. bimodal) singularity, and f is good if and only if

$$\delta_2(X, x) \geq 2.$$

(2.5) QUESTION. Does the inequality $\delta_2 \geq m(X, x)$ ($m(X, x)$ denotes the modality of a hypersurface singularity) hold?

That holds for quasi-homogeneous hypersurface singularities (see [7]).

3 The equisingular deformations

We follow the notation and terminology of the second section. We always assume that (X, x) is a minimally elliptic singularity.

Let ES be the equisingular deformation functor in the sense of [3]. By [2], an equisingular deformation of M induces a topologically constant deformation of a singularity (X, x) .

By [3], ES is smooth and the tangent space of ES is $H^1(M, S)$, where $S = \mathcal{H}_{\mathcal{O}_M/\mathcal{O}_M}(\Omega_M^1(\log E), \mathcal{O}_M)$ (which is a locally free sheaf of rank 2).

(3.1) DEFINITION. We define an invariant $q(X, x)$ called *irregularity* by

$$q(X, x) = \dim_{\mathbb{C}} H^0(M - E, \Omega_{M-E}^1) / H^0(M, \Omega_M^1).$$

If (X, x) is a simple elliptic or not a quasi-homogeneous singularity, then $q(X, x) = 0$, and $q(X, x) = 1$ for every other (cf. [4, THEOREM 1.9]).

We denote by $h^i(\cdot)$ (resp. $h_E^1(\cdot)$) the dimension of \mathbb{C} -vector space $H^i(\cdot)$ (resp. $H_E^1(\cdot)$).

(3.2) PROPOSITION. *If (X, x) is of general type, then*

$$h^1(S) = q(X, x) + h^0(D, \Omega_M^1(\log E) \otimes \mathcal{O}_D(-E)),$$

where $D = -K - E$.

Proof. By the duality, $h^1(S) = h_E^1(\Omega_M^1(\log E) \otimes \mathcal{O}(K))$. By Wahl's vanishing theorem, $h^1(\Omega_M^1(\log E) \otimes \mathcal{O}(K)) = 0$. Hence we have

$$h_E^1(\Omega_M^1(\log E) \otimes \mathcal{O}(K)) = \dim_{\mathbb{C}} H^0(M-E, \Omega_{M-E}^1) / H^0(\Omega_M^1(\log E) \otimes \mathcal{O}(K))$$

and

$$h^0(D, \Omega_M^1(\log E) \otimes \mathcal{O}_D(-E)) = \dim_{\mathbb{C}} H^0(\Omega_M^1(\log E) \otimes \mathcal{O}(-E)) / H^0(\Omega_M^1(\log E) \otimes \mathcal{O}(K)).$$

From the exact sequence

$$0 \rightarrow H^0(\Omega_M^1(\log E) \otimes \mathcal{O}(-E)) \rightarrow H^0(\Omega_M^1) \rightarrow H^0(\oplus \Omega_{E_i}^1) = 0,$$

and inclusions

$$H^0(\Omega_M^1(\log E) \otimes \mathcal{O}(K)) \subseteq H^0(\Omega_M^1(\log E) \otimes \mathcal{O}(-E)) \subseteq H^0(\Omega_M^1) \subseteq H^0(M-E, \Omega_M^1),$$

we have the assertion of the proposition.

In a similar way as above, we have $h^1(S) = 1$ for simple elliptic singularities and $h^1(S) = 0$ for cusp singularities.

(3.3) EXAMPLE. Let (X, x) be of general type and $E = \bigcup_{i=1}^r E_i$ the irreducible decomposition. Assume that dual graph of E is star-shaped such that $E_0 E_i = 1$ for $i = 1, \dots, r$, $E_i E_j = 0$ if $1 \leq i < j \leq r$ and $E_0^2 = -r + 2$.

Then $K = -2E_0 - \sum_{i=1}^r E_i$, and $D = -K - E = E_0$. Hence $\delta_2(X, x) = KD + 2 = r - 2$.

There is an isomorphism

$$\Omega_M^1(\log E) \otimes \mathcal{O}_{E_0}(-E) \cong \mathcal{O}_{E_0}(-4 + r) \oplus \mathcal{O}_{E_0}(-2).$$

By (3.2), $h^1(S) = q(X, x) + r - 3$. Hence $\delta_2(X, x) = h^1(S) - q(X, x) + 1$.

If (X, x) is a quasi-homogeneous hypersurface singularity, and moreover if the invariance of Milnor's number implies the invariance of the topological type, then $\delta_2(X, x) = m(X, x)$.

By [4, (1.5), (1.6)], the exterior differentiation gives an exact sequence

$$0 \rightarrow d\mathcal{O}(K) \rightarrow \Omega_M^1(\log E) \otimes \mathcal{O}(K) \rightarrow \Omega_M^2 \otimes \mathcal{O}(K + E) \rightarrow 0$$

and isomorphisms $H^i(\mathcal{O}(K)) \cong H^i(d\mathcal{O}(K))$ for all i , where we consider $\mathcal{O}(K)$ an ideal sheaf of \mathcal{O}_M .

There is an exact sequence

$$\begin{aligned} 0 \rightarrow H^0(d\mathcal{O}(K)) \rightarrow H^0(M - E, d\mathcal{O}) \rightarrow H_E^1(d\mathcal{O}(K)) \\ \rightarrow H^1(d\mathcal{O}(K)) \cong H^1(\mathcal{O}(K)) = 0. \end{aligned}$$

If (X, x) is of general type, then $H_{\{x\}}^1(d\mathcal{O}_X) = 0$ by [4, (1.13.4)]. Hence

$H^0(X, d\mathcal{O}_X) \cong H^0(M - E, d\mathcal{O})$, and the map $H^0(d\mathcal{O}(K)) \rightarrow H^0(M - E, d\mathcal{O})$ is surjective. Hence $H_E^1(d\mathcal{O}(K)) = 0$.

By the exact sequence

$$\begin{aligned} H_E^1(d\mathcal{O}(K)) &\rightarrow H_E^1(\Omega_M^1(\log E) \otimes \mathcal{O}(K)) \rightarrow H_E^1(\mathcal{O}(2K + E)) \\ &\rightarrow H_E^2(d\mathcal{O}(K)) \rightarrow H_E^2(\Omega_M^1(\log E) \otimes \mathcal{O}(K)), \end{aligned}$$

we have $\delta_2(X, x) = h^1(S) + \alpha$ for the singularity (X, x) of general type, where

$$\alpha = \dim_{\mathbb{C}} \text{Ker}(H_E^2(d\mathcal{O}(K)) \rightarrow H_E^2(\Omega_M^1(\log E) \otimes \mathcal{O}(K))).$$

REFERENCES

1. Ishii, S.: The asymptotic behavior of plurigenera for a normal isolated singularity. Math. Ann. 286, 803-812(1990)
2. Laufer, H.B.: Weak simultaneous resolution for deformations of Gorenstein surface singularities. Proc. Sympos. Pure Math., vol 40, Part 2, 1-29(1983)
3. Wahl, J.M.: Equisingular deformations of normal surface singularities, I. Ann. of Math. 104, 325-356(1976)
4. Wahl, J.M.: A characterization of quasi-homogeneous Gorenstein singularities. Composito Math. 55, 269-288(1985)
5. Watanabe, K.: On plurigenera of normal isolated singularities I. Math. Ann. 250, 65-94(1980)

6. Watanabe,K., Okuma,T.: Characterization of unimodular singularities and bimodular singularities by the second plurigenus. Preprint.

7. Yoshinaga,E., Watanabe,K.: On the geometric genus and the inner modality of quasihomogeneous isolated singularities. Sci. Rep. Yokohama Nat. Univ. Sect. I 25, 45-53(1978)